§1) Semi-Riemannian Mflds

Defn: (Semi-Riemm. mfld)

A semi-Riemannian mfld (M,g) consists of:

1) M, a smooth n-dim mfld
2) Tensor field, g s.t. $\forall a \in M$ it assigns a non-deg. & symm. bilinear form on T_aM:

$$g_a : T_aM \times T_aM \longrightarrow \mathbb{R}$$

Example:

1) $\mathbb{R}^{p,q} = (\mathbb{R}^{p+q}, g^{p,q})$ for $p,q \in \mathbb{N}$ s.t.

$$g^{p,q}(\underbrace{X, Y}_{T_a\mathbb{R}^{p,q}}) := \sum_{i=1}^{p} x^i y^i - \sum_{i=p+1}^{q} x^i y^i$$

(or) $$(g_{\mu\nu}) = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix}$$

2) $\mathbb{R}^{1,3}$ (or) $\mathbb{R}^{3,1}$ Minkowski space

§2) Conformal Transformations

Defn: (Conformal transformation)

(M,g) and (M',g') be two semi-Riemannian mflds of dim n, $U \subset M$, $V \subset M'$ open sets. A smooth map $\varphi : U \longrightarrow V$ of maximal rank is called a conformal transformation if $\exists$ a smooth fn. $\Omega : U \to \mathbb{R}_+$ s.t.

$$\varphi^* g' = \Omega^2 g$$

$\varphi^* g'(X,Y) = g'(T\varphi(X), T\varphi(Y))$ where $T\varphi : TU \to TV$ denotes the tangent map of φ (derivative)

The smooth fn Ω is called conformal factor of φ

Remarks:

1) In local coordinates of M & M', φ is conformal iff:

$$(g'_{ij} \circ \varphi)\, \partial_\mu \varphi^i \partial_\nu \varphi^j = \Omega^2 g_{\mu\nu}$$

2) The maps $T_a\varphi : T_a M \to T_{\varphi(a)} M'$ are bijective $\forall a \in M$ if φ is conformal. Then inverse mapping thm $\Rightarrow$ φ is a local diffeomorphism.

Examples:

1) Local isometries (or smooth maps φ with $\varphi^* g' = g$) are trivial conformal transformations with $\Omega = 1$

2) A smooth map $\varphi : \overset{\mathbb{C}}{\overset{\cup}{M}} \to \mathbb{C}$ (where M is a connected open subset of $\mathbb{C}$) is conformal with $\Omega : M \to \mathbb{R}_+$ if & only if

- $u_x^2 + v_x^2 = \Omega^2 = u_y^2 + v_y^2 \neq 0$ $\quad \left(\begin{array}{l} u = \operatorname{Re}\varphi \; ; \; u_x \text{ derivative wrt } x \\ v = \operatorname{im}\varphi \quad\quad \text{etc} \end{array} \right)$

- $u_x u_y + v_x v_y = 0$

But Holomorphic & antiholomorphic functions (which satisfy Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ (or) "anti"- Cauchy-Riemann eqns $u_x = -v_y$, $u_y = v_x$) from $M \to \mathbb{C}$ with $u_x^2 + v_x^2 \neq 0$ satisfy the above two conditions. For holomorphic (or) antiholomorphic functions
$u_x^2 + v_x^2 \neq 0 \iff \det \underbrace{D\varphi}_{} \neq 0$

$D\varphi$ = Jacobi matrix of $T\varphi$

Overall,

$$\left(\text{Conf. transf. } \varphi : M \to \mathbb{C} \right) \Leftarrow \left(\begin{array}{l} \text{locally invertible} \\ \text{holomorphic / antiholomorphic} \\ \text{functions} \end{array} \right)$$

$\Rightarrow$ direction is also true:

For a general conf. tran. $\varphi = (u, v)$, the above two conditions imply (u_x, v_x) and (u_y, v_y) are "perpendicular" vectors in $\mathbb{R}^{2,0}$ of length $\Omega^2 \neq 0$ ie)

$$(u_x, v_x) = (-v_y, u_y) \text{ (or) } (u_x, v_x) = (v_y, -u_y)$$

meaning φ is holomorphic / anti-holomorphic with $\det D\varphi \neq 0$

So, $$\left(\text{Conf. transf. } \varphi : M \to \mathbb{C} \right) \iff \left(\begin{array}{l} \text{locally invertible} \\ \text{holomorphic / antiholomorphic} \\ \text{functions} \end{array} \right)$$

3) In what sense, conformal transformations "preserve angles"?:
$\mathbb{C} \cong \mathbb{R}^{2,0}$, a linear map $\varphi : \mathbb{R}^{2,0} \to \mathbb{R}^{2,0}$ with matrix rep.

$$A_\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{(Jacobian of } \varphi)$$

is conformal if & only if (following in close analogy to 2) above) $(a^2+c^2 \neq 0,\ a=d,\ b=-c)$ (or) $(a^2+c^2 \neq 0, a=-d, b=c)$. Substituting these conditions in A_φ and making it act on a complex no. $z = x+iy$, one can observe the φ acts in the following form(s):

$$z \longmapsto \xi z \quad \text{(or)} \quad z \longmapsto \xi \bar{z} \quad \text{where } \xi = a+ic \neq 0$$

Then, $\omega(z,w) := \frac{z\bar{w}}{|zw|}$ determines "angle" between z and w up to orientation. So in our case, it follows by direct calculation that:

$$\omega(\varphi(z), \varphi(w)) = \frac{\xi z\, \overline{\xi w}}{|\xi z\, \xi w|} = \frac{z\bar{w}}{|zw|} = \omega(z,w)$$

$$\varphi : \mathbb{R}^{2,0} \to \mathbb{R}^{2,0}$$
$$z \mapsto \xi z$$

(or) when $\varphi : z \longmapsto \xi \bar{z}$, one can show the same thing holds.

Conversely: linear maps φ with $\omega(\varphi(z), \varphi(w)) = \omega(z,w)$ $\forall z,w \in \mathbb{C}\setminus\{0\}$

(or) $\omega(\varphi(z), \varphi(w)) = -\omega(z,w) \ \forall z,w \in \mathbb{C}\setminus\{0\}$

are conformal transformations.

Overall, $\mathbb{R}$-linear maps $\varphi : \mathbb{R}^{2,0} \to \mathbb{R}^{2,0}$ are

$$(\text{Conformal}) \iff (\text{Angle preserving in the above sense})$$

§3) Conformal Killing Fields

Goal: Study conformal maps $\varphi : M \to M'$ where $M, M' \subset \mathbb{R}^{p,q}$ with $p+q =: n > 1$

Defn (Local flow & local one parameter group)

X is any C^∞-vector field on a chart U & $p \in U$, then there are neighborhood W of $p \in U$, an $\varepsilon > 0$ and a C^∞ map

$$\varphi^X : (-\varepsilon, \varepsilon) \times W \longrightarrow U$$

s.t. for each $a \in W$, the fn $\varphi^X(t,a)$ is an integral curve of X starting at a. ie) $\varphi^X(0,a) = a$ and if also satisfies the following "group-like" behaviour

$$\varphi^X_t(\varphi^X_s(a)) = \varphi^X_{t+s}(a) \quad (\text{wherever they are defined})$$

φ^X is called <u>local flow generated by a vector field, X</u>

Why do we need local flow?

Because,

$\varphi^X(0,a) = a$ and $\underbrace{\frac{\partial \varphi^X}{\partial t}(t,a) = X_{\varphi^X(t,a)}}$
$\forall a \in W \subset U$

φ^X satisfies the "flow equation"

At $t=0$,

$$\frac{\partial \varphi^X(0,a)}{\partial t} = X_{\underbrace{\varphi^X(0,a)}_{=a}} = X_a$$

ie) One can recover the whole vector field from its local flows. Denote $\varphi(t,\cdot)$ as $\varphi_t(\cdot)$, then,

$\{\varphi_t^X\}_{t\in\mathbb{R}}$ is called the local one-parameter group of X

<u>Defn: (Conformal Killing field)</u>

A vector field X on $M \subset \mathbb{R}^{p,q}$ is called <u>conformal</u> if φ_t^X is conformal for all t in the neighborhood of 0.

<u>Thm: *)</u>

Let $M \subset \mathbb{R}^{p,q}$ and $g = g^{p,q}$ (for Minkowski) and X be a conformal field. Then $X = (X^1, \dots, X^n) = X^\xi \partial_\xi$

wrt canonical coord. of $\mathbb{R}^n$. Then $\exists$ a smooth fn. $\kappa : M \to \mathbb{R}$ s.t.

$$\underbrace{\partial_\nu X_\mu}_{} + \partial_\mu X_\nu = \kappa\, g_{\mu\nu}$$

where

$$\partial_\nu X_\mu = \partial_\nu\, g_{\mu\sigma} X^\sigma$$

Proof:

X is conformal killing field. $(\varphi_t)_{t\in\mathbb{R}}$, $\Omega_t : M_t \to \mathbb{R}^+$ s.t.

$$(\varphi_t^* g)_{\mu\nu}(a) = \underbrace{\underbrace{g_{ij}(\varphi_t(a))}_{\text{constant} \,\because\, \text{we're in } \mathbb{R}^{p,q}}\, \partial_\mu \varphi_t^i\, \partial_\nu \varphi_t^j}_{} = (\Omega_t(a))^2 g_{\mu\nu}(a)$$

Differentiate wrt t on both sides and restrict to $\big|_{t=0}$:

$$g_{ij}\, \partial_\mu \dot{\varphi}_0^i(a)\, \underbrace{\partial_\nu \varphi_0^j(a)}_{=\delta_\nu^j} + g_{ij}\, \partial_\mu \varphi_0^i(a)\, \partial_\nu \dot{\varphi}_0^j(a) = \frac{d}{dt}\left((\Omega_t(a))^2\right)\Big|_{t=0} g_{\mu\nu}(a)$$

Recall φ_0 is identity ($\because \varphi$ is a flow of X)
So, Jacobian of identity is just Kronecker delta coordinate-wise

$$\Rightarrow g_{ij}\, \partial_\mu X^i(a)\, \delta_\nu^j + g_{ij}\, \delta_\mu^i\, \partial_\nu X^j(a) = \frac{d}{dt}\left((\Omega_t(a))^2\right)\Big|_{t=0} g_{\mu\nu}(a)$$

ii) $\partial_\mu X_\nu + \partial_\nu X_\mu = \underbrace{\frac{d}{dt}\left(\left(\Omega_t(a)\right)^2\right)\Big|_{t=0}}_{=:\kappa(a)} g_{\mu\nu}(a)$

"infinitesimal" conformal tran.

The above statement motivates the following definition:

Defn: (Conformal Killing factor)

Let $\kappa: \underset{\mathbb{R}^{p,q}}{\underset{\cap}{M}} \to \mathbb{R}$ be a smooth function. κ is called a conformal killing factor if there exists a conformal killing field X s.t.

$$\partial_\mu X_\nu + \partial_\nu X_\mu = \kappa g_{\mu\nu}$$

where $X_\nu = g_{\nu\sigma} X^\sigma$

Theorem:

$$(\kappa: M \to \mathbb{R} \text{ is a conformal killing factor})$$

$$\Updownarrow$$

$$(n-2)\partial_\mu \partial_\nu \kappa + g_{\mu\nu} \underbrace{g^{kl} \partial_k \partial_l}_{\substack{\shortparallel \\ \Delta_g}} \kappa = 0$$

Proof:

Proof uses index gymnastics which is not too interesting to see. ($\Rightarrow$ of proof:)

(Thm-1.6 in main reference [S] pages 14, 15)

$\Leftarrow$-Direction follows in §4) below

Remarks:

$$\begin{pmatrix} K \text{ is a conformal} \\ \text{Killing factor} \end{pmatrix} \Leftrightarrow (n-2)\partial_\xi \partial_\nu K + g_{\xi\nu} \underbrace{g^{kl}\partial_k \partial_l K}_{\overset{!!}{=}\,\Delta_g} = 0$$

For $n=2$, this means

$$\Leftrightarrow \Delta_g K = 0$$

For $n>2$, K is a (conf. Killing factor) $\Leftrightarrow$:

- For $\xi \neq \nu$, $\partial_\xi \partial_\nu K = 0$
- For $\xi = \nu$, $(n-2)\partial_\xi^2 K + \overset{\pm 1}{g_{\xi\xi}} \Delta_g K = 0$

  (or) $\partial_\xi^2 K = \dfrac{\pm 1}{(n-2)} \Delta_g K$

§4) Classification of Conf. transf.

Conf. tran. of connected open sets of $M \subset \mathbb{R}^{p,q}$

Case-1 $p+q =: n > 2$

conf. Kill factor $\Rightarrow$ $(n-2)\partial_\xi \partial_\nu K + g_{\xi\nu}\Delta_g K = 0$

For $\eta = \nu$:

$$(n-2)\, g_{\xi\xi}\, \partial_\xi \partial_\xi \zeta + \Delta_g \zeta = 0$$

$\sum_\xi$ on both sides,

$$(n-2)\Delta_g \zeta + \Delta_g \zeta = 0$$

(or) $\Delta_g \zeta = 0$. Substitute this back in to get:

$$\partial_\xi^2 \zeta = 0 \implies \partial_\xi \zeta(q^1, \dots, q^n) = \underset{(\text{const.})}{\alpha_\xi \in \mathbb{R}} \quad \xi = 1, \dots, n$$

$$\Downarrow$$

$$\zeta(q) = \lambda + \alpha_\nu q^\nu \quad q = (q^\nu) \in M \subset \mathbb{R}^n$$

$$(\lambda, \alpha_\nu \in \mathbb{R})$$

- <u>Case-1 a)</u> $\zeta = 0 \longrightarrow$ exponentiating $\Omega = 1$ aka local isometries

So, $\partial_\xi X_\eta + \partial_\eta X_\xi = 0$

Similarly, $\partial_\nu X_\eta + \partial_\eta X_\nu = 0$

$$\implies X^\eta(q) = c^\xi + \omega^\xi_\nu q^\nu \quad \text{where, } c^\xi, \omega^\xi_\nu \in \mathbb{R}$$

- If ω^ξ_ν vanish: $X^\eta(q) = c^\xi$

$\rightsquigarrow \dot{q} = c$ Solve

$\Rightarrow \varphi^X(t,q) = q + tc$ is the Global one-parameter group

Associated conformal trans. $(t=1)$ $\rightsquigarrow \varphi^X(q) = q + c \longrightarrow$ Translation!

- If $c = 0$, $\omega = (\omega^\xi_\nu)$, then

$$X_{\xi,\nu} + X_{\nu,\xi} = g_{\xi\nu} \zeta = 0$$

leads to

$$g_{\nu\beta}\omega^\beta_\xi + g_{\xi\beta}\omega^\beta_\nu = 0$$

ie) $\omega^T g + g\omega = 0$ ie) elements of $o(p,q) = \{\omega \mid \omega^T g + g\omega = 0\}$

Global one parameter flow: $\varphi^X(t,q) = e^{t\omega} q$

For $t=1$, $\varphi(q) = e^{\omega} q$ being the associated conformal tran.

ie) $\varphi_\Lambda : \mathbb{R}^{p,q} \longrightarrow \mathbb{R}^{p,q}$

$(q \in \mathbb{R}^n)$ $\quad q \longmapsto \Lambda q$ where

$$\Lambda \in O(p,q) = \{\Lambda \in \mathbb{R}^{n \times n} \mid \Lambda^T g^{p,q} \Lambda = g^{p,q}\}$$

$\rightsquigarrow$ ie) $\langle \Lambda x, \Lambda x' \rangle = \langle x, x' \rangle$

<u>Case-1b):</u> $\kappa = \lambda \in \mathbb{R} \setminus \{0\}$ const.

$$\rightsquigarrow \quad X(q) = \lambda q$$

$$\rightsquigarrow$$

Associated conf. tran. $\varphi(q) = e^{\lambda} q$, $q \in \mathbb{R}^n$ Dilations.

<u>Case-1 c)</u> $\kappa \neq 0$

$\kappa(q) = \lambda + \alpha_\nu q^\nu$ If $\lambda = 0$, then,

Parametrize as: $\kappa(q) = 4\langle q, b\rangle$ $q \in \mathbb{R}^n$, $b \in \mathbb{R}^n \setminus \{0\}$

$$\rightsquigarrow$$

(Has no global one-parameter group) $\Leftarrow$ $X^\mu(q) := 2\langle q, b\rangle q^\mu - \langle q, q\rangle b^\mu$, $\begin{pmatrix} q \in \mathbb{R}^n \\ b \in \mathbb{R}^n \setminus \{0\} \end{pmatrix}$

is the soln. of $\partial_\nu X_\mu + \partial_\mu X_\nu = \kappa g_{\mu\nu}$

Thus proving "$\Leftarrow$" direction of Thm*) in pg 6/7 of these notes.

So, for every conformal Killing field X with conformal Killing factor $\kappa(q) = \lambda + \alpha_\mu q^\mu = \lambda + 4\langle q, b\rangle$, we can

define a vector field $Y(q) = X(q) - 2\langle q,b\rangle q + \langle q,q\rangle b - \lambda q$
which is a conf. Killing field with conformal Killing factor $\xi = 0$.

$\Rightarrow$ $Y(q) = c + \omega q$ by Case-i) discussion above.

ie) We proved:

Thm:

Every conformal Killing field X on a connected open subset M of $\mathbb{R}^{p,q}$ (for $p+q =: n > 2$) is of the form:

$$X(q) = 2\langle q,b\rangle q - \langle q,q\rangle b + \lambda q + c + \omega q$$

for suitable $b, c \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, $\omega \in \mathfrak{o}(p,q)$

The conformal Killing field

$$X(q) = 2\langle q,b\rangle q - \langle q,q\rangle b, \quad b \neq 0, \quad q \in \mathbb{R}^n$$

we described above has NO global one-parameter grp of solutions for $\dot{q} = X(q)$. Its solutions form the local one parameter group

$$\varphi_t(q) = \frac{q - \langle q,q\rangle tb}{1 - 2\langle q,tb\rangle + \langle q,q\rangle\langle tb,tb\rangle}, \quad t \in (t_q^-, t_q^+)$$

where (t_q^-, t_q^+) is the maximal interval around 0 contained in

$$\{t \in \mathbb{R} \mid 1 - 2\langle q, tb\rangle + \langle q,q\rangle\langle tb, tb\rangle \neq 0\}$$

So, the associated conformal transformation $(t=1)$ is:

$$\varphi(q) = \frac{q - \langle q,q\rangle b}{1 - 2\langle b,q\rangle + \langle q,q\rangle\langle b,b\rangle}$$

which are called special conformal transformations

Summary:

Every conformal transf. $\varphi: M \to \mathbb{R}^{p,q}$ for $p+q>2$, (M being a connected open subset of $\mathbb{R}^{p,q}$) is a composition of the following four types of maps

1) a translation $q \longmapsto q+c$, $c \in \mathbb{R}^n$

2) an orthogonal transformation, $q \longmapsto \Lambda q$, $\Lambda \in O(p,q)$

3) a dilation, $q \longmapsto e^{\lambda} q$, $\lambda \in \mathbb{R}$

4) a special conformal transformation

$$q \longmapsto \frac{q - \langle q,q\rangle b}{1 - 2\langle q,b\rangle + \langle q,q\rangle\langle b,b\rangle}, \quad b \in \mathbb{R}^n.$$

Case-2): Euclidean plane $p=2, q=0$

Already discussed in Example-2 of examples of conformal transformations ie) Stated again:

Case-3: Minkowski plane ($p=q=1$)

Thm: A smooth map $\varphi = (u,v) : M \to \mathbb{R}^{1,1}$ on a connected open subset of $M \subset \mathbb{R}^{1,1}$ is conformal if & only if:

$$u_x^2 > v_x^2 \quad \text{and} \quad (u_x = v_y,\ u_y = v_x)$$

$$\text{(or)}$$

$$(u_x = -v_y,\ u_y = -v_x)$$

Corollary: The orientation-preserving linear & conformal maps $\psi : \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$ have matrix representations of the form

$$A = A_\psi = \exp t \begin{pmatrix} \cosh(s) & \sinh(s) \\ \sinh(s) & \cosh(s) \end{pmatrix}$$

$$\text{(or)}$$

$$= \exp t \begin{pmatrix} -\cosh(s) & \sinh(s) \\ \sinh(s) & -\cosh(s) \end{pmatrix}$$

with $(s,t) \in \mathbb{R}^2$.

Interpret t = dilation & s boost similar to Euclidean case seen above.